

WEAK INSTABILITY OF HAMILTONIAN EQUILIBRIA

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ABSTRACT. This is an expository paper on Lyapunov stability of equilibria of autonomous Hamiltonian systems. Our aim is to clarify the concept of *weak instability*, namely instability without non-constant motions which have the equilibrium as limit point as time goes to minus infinity. This is done by means of some examples. In particular, we show that a weakly unstable equilibrium point can be stable for the linearized vector field.

1. INTRODUCTION

Stability of the equilibrium is a mathematical field more than two centuries old. Indeed, Lagrange stated the celebrated Lagrange-Dirichlet theorem in the eighteenth century, and some so called converses of that statement are still proved nowadays. So many mathematicians have been interested in stability that we refrain from mentioning them with the exception of the most important, Lyapunov, who defended his doctoral thesis “The general problem of the stability of motion” in 1892. The applications are also countless in mechanics and in most sciences. To start with the rich literature on this matter, see Arnold et al. [1], Meyer et al. [4], and Rouche et al. [7].

Important mathematical objects related to the instability of the equilibrium are *asymptotic motions*. Before their formal definition, let us mention that the upper position of a simple pendulum, and zero velocity, constitute an unstable equilibrium and its asymptotic motions are neither rotations (when the pendulum swings around and around) nor librations (when it swings back and forth), and they stay between the two behaviors.

Let us consider a smooth vector field f on an open set $A \subseteq \mathbb{R}^N$ with an equilibrium point $\hat{x} \in A$, so $f(\hat{x}) = 0$. We say that $\phi : (-\infty, b) \rightarrow A$ is an *asymptotic motion* in the past to the equilibrium point \hat{x} , if $\phi(t)$ is a non-constant solution to the o.d.e. $\dot{x} = f(x)$ such that $\phi(t) \rightarrow \hat{x}$ as $t \rightarrow -\infty$. In the sequel we briefly write ‘asymptotic motion’ instead of ‘asymptotic motion in the past’ since we are only concerned with this kind of asymptotic motions. Of course the existence of an asymptotic motion implies the Lyapunov instability of the equilibrium point. The

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basic sufficient condition for the existence of an asymptotic motion is the presence of an eigenvalue of $f'(\hat{x})$ with strictly positive real part, see for instance Hartman [3] remark to Corollary 6.1, p. 243.

In this paper we focus on *autonomous Hamiltonian systems* so in the sequel $N = 2n$, $x = (q, p)$, $q, p \in \mathbb{R}^n$, and the vector field is

$$(1) \quad (\partial_p H(q, p), -\partial_q H(q, p))$$

for some smooth H called the Hamiltonian function. Our aim is to clarify the concept of *Lyapunov instability without asymptotic motions* that we briefly call *weak instability*. This is done by means of some examples.

Section 2 deals with linear systems. Of course there is a trivial situation where weak instability appears: the free particle. The equilibrium is non-isolated and the eigenvalues vanish, the example can be done in one degree of freedom so in dimension 2. A more subtle instability of the equilibrium for a linear system is obtained when the eigenvalues are purely imaginary and some Jordan blocks have dimension greater than one, of course this can happen only in dimension at least 4. The example we are going to see comes from the planar restricted 3-body problem at one of the relative equilibria, the Lagrange equilateral points, also called the Trojan points, at the critical Routh value of the mass ratio of the primaries.

In Section 3 we move on nonlinear systems. Their equilibria can be unstable even if we have stability for the linearized system as the Cherry Hamiltonian in dimension 4 shows by means of an asymptotic motion. Cherry's system is the third example of this paper, it was published in 1925 and, in the last 20 years, it became important in plasma physics, see Pfirsch [6] and the references therein.

Our fourth example, also in dimension 4, comes from [11] and shows that *we can have weak instability of an Hamiltonian equilibrium which is linearly stable*. Some systems, produced by Barone-Netto and myself [9] and [10], preceded [11], they give non-Hamiltonian examples of weak instability for linearly stable equilibria.

Hopefully, the concept of weak instability will stimulate further researches in stability within mathematical physics, together with other fresh notions like the "weak asymptotic stability" introduced by Ortega, Planas-Bielsa and Ratiu, see [5] and the references therein.

2. WEAK INSTABILITY FOR LINEAR SYSTEMS

2.1. Free particle. Our first example is a particle on a straight line under no forces

$$(2) \quad H(q, p) = \frac{p^2}{2}, \quad q, p \in \mathbb{R}.$$

The Hamiltonian vector field is

$$(3) \quad (\partial_p H(q, p), -\partial_q H(q, p)) = (p, 0).$$

It is a linear field with the double eigenvalue 0. The integral curves are

$$(4) \quad q(t) = q(0) + p(0)t, \quad p(t) = p(0).$$

Each $(q_0, 0) \in \mathbb{R}^2$ is an equilibrium point and its instability can be shown by means of the sequence $(q(0), p(0)) = (q_0, 1/m) \rightarrow (q_0, 0)$ as $m \rightarrow +\infty$. There are no asymptotic motions.

2.2. Linearization at L_4 . Our second example is the quadratic part of the Hamiltonian function of the planar restricted 3-body problem at one of the relative equilibria, the Lagrange libration point L_4 at the critical Routh value of the mass ratio of the primaries. In the sequel $q = (q_1, q_2), p = (p_1, p_2), (q, p) = (q_1, q_2, p_1, p_2)$, and

$$(5) \quad H(q, p) = \frac{1}{\sqrt{2}} \det(p, q) + \frac{1}{2} |q|^2 = \frac{1}{\sqrt{2}} (p_1 q_2 - p_2 q_1) + \frac{1}{2} (q_1^2 + q_2^2).$$

The Hamiltonian vector field is

$$(6) \quad \begin{aligned} &(\partial_{p_1} H(q, p), \partial_{p_2} H(q, p), -\partial_{q_1} H(q, p), -\partial_{q_2} H(q, p)) = \\ &= \left(q_2/\sqrt{2}, -q_1/\sqrt{2}, -q_1 + p_2/\sqrt{2}, -q_2 - p_1/\sqrt{2} \right), \end{aligned}$$

see H_0 and the o.d.e. at the end of p. 256, with $\xi = q, \eta = p, \omega = 1/\sqrt{2}, \delta = 1$, and also H_0 at p. 258 in Meyer et al. [4].

It is a linear vector field with the double eigenvalues $\lambda = \pm i/\sqrt{2}$ and Jordan blocks $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. The origin is now the unique equilibrium point.

The function $|q|^2$ is a first integral. Suppose the integral curve $(q(t), p(t)) \rightarrow 0$ as $t \rightarrow -\infty$, then $|q(t)|^2 \equiv 0$ and this fact further implies that $|p(t)|^2 \equiv 0$, indeed for $q(t) \equiv 0$ we have

$$(7) \quad \frac{d}{dt} |p(t)|^2 = 2p(t) \cdot \left(-q_1(t) + p_2(t)/\sqrt{2}, -q_2(t) - p_1(t)/\sqrt{2} \right) = 0.$$

So the integral curve is constant and we do not have asymptotic motions to the equilibrium point.

The origin is an unstable equilibrium point as we can see with

$$(8) \quad \begin{aligned} q_1(t) &= \frac{1}{m} \cos \frac{t}{\sqrt{2}}, & q_2(t) &= -\frac{1}{m} \sin \frac{t}{\sqrt{2}}, \\ p_1(t) &= -\frac{t}{m} \cos \frac{t}{\sqrt{2}}, & p_2(t) &= \frac{t}{m} \sin \frac{t}{\sqrt{2}}, \end{aligned}$$

for $w = (q_1(0), q_2(0), p_1(0), p_2(0)) = (1/m, 0, 0, 0) \rightarrow 0$ as $m \rightarrow +\infty$.

Incidentally, in connection with the nonlinear 3-body problem which has the Hamiltonian vector field defined by (6) as linearization at L_4 , the book [4] at the end of Sec. 13.6 says that in 1977 two papers claimed to have proved the stability of the equilibrium, however one proof is

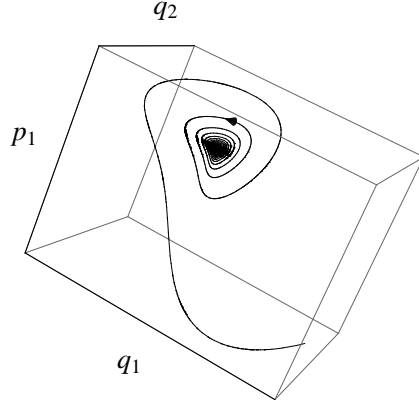


FIGURE 1. Asymptotic motion for Cherry Hamiltonian

wrong and the other is unconvincing. The last sentence is: “It would be interesting to give a correct proof of stability in this case, because the linearized system is not simple, and so the linearized equations are unstable”.

3. INSTABILITY FOR LINEARLY STABLE EQUILIBRIA

3.1. Cherry Hamiltonian. Next, the famous Cherry Hamiltonian system shows that the equilibrium can be unstable even if it is stable for the linearized system, briefly even if it is *linearly stable*. In Cherry [2] p. 199, or in Whittaker [8] p. 412, we can see the Hamiltonian function $H : \mathbb{R}^4 \rightarrow \mathbb{R}$

$$(9) \quad H(q, p) = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) + \sigma(q_2(q_1^2 - p_1^2) - 2q_1p_1p_2).$$

The Hamiltonian vector field, written as a column vector, is

$$(10) \quad \begin{pmatrix} p_1 - 2\sigma q_2 p_1 - 2\sigma q_1 p_2 \\ -2p_2 - 2\sigma q_1 p_1 \\ -q_1 - 2\sigma q_2 q_1 + 2\sigma p_1 p_2 \\ 2q_2 + \sigma p_1^2 - \sigma q_1^2 \end{pmatrix}.$$

The linearized vector field $(p_1, -2p_2, -q_1, 2q_2)$ is obtained for $\sigma = 0$. The origin is stable for the linearized systems which consists of two harmonic oscillators: $\ddot{q}_1 = -q_1$, $\ddot{q}_2 = -4q_2$. The eigenvalues are distinct $\pm i$, $\pm 2i$. However, the origin is Lyapunov unstable for the vector field (10) whenever $\sigma \neq 0$ since it has the following asymptotic motion defined for $t < 0$

$$(11) \quad \begin{aligned} q_1(t) &= \frac{\sin t}{\sqrt{2}\sigma t}, & q_2(t) &= \frac{\sin(2t)}{2\sigma t}, \\ p_1(t) &= \frac{\cos t}{\sqrt{2}\sigma t}, & p_2(t) &= -\frac{\cos(2t)}{2\sigma t}. \end{aligned}$$

3.2. Variation-like Hamiltonian. Our final example shows that the origin is an unstable equilibrium point which is linearly stable and has no asymptotic motions for the system defined by

$$(12) \quad H(q, p) = p_1 p_2 + q_1 q_2 + \sigma q_1^2 q_2, \quad \sigma \neq 0.$$

It is a particular case of the following Hamiltonian function introduced in [11]

$$(13) \quad H(q, p) = p_1 p_2 + g(q_1) q_2, \quad g(0) = 0, \quad g'(0) > 0,$$

where $g \in C^1$ on a neighborhood of 0. The Hamiltonian vector field is

$$(14) \quad \begin{pmatrix} p_2 \\ p_1 \\ -g'(q_1) q_2 \\ -g(q_1) \end{pmatrix} = \begin{pmatrix} p_2 \\ p_1 \\ -g'(0) q_2 \\ -g'(0) q_1 \end{pmatrix} + o(|(q, p)|).$$

The origin is stable for the linearized system which consists of two harmonic oscillators: $\ddot{q}_1 = -g'(0) q_1$, $\ddot{q}_2 = -g'(0) q_2$. In this case the eigenvalues are double $\pm i\sqrt{g'(0)}$ however the Jordan blocks are one-dimensional.

The subsystem of the first and last canonical equations

$$(15) \quad \dot{q}_1 = p_2, \quad \dot{p}_2 = -g(q_1),$$

separates. If we take a solution $(q_1(t), p_2(t))$ of this subsystem and plug $q_1(t)$ into the second and third canonical equations, we then get the equations of variation of (15) along the solution $(q_1(t), p_2(t))$. This is why the function in formula (13) is called *variation-like Hamiltonian* in the title of this subsection.

There are no asymptotic motions, indeed if the solution

$$(16) \quad (q_1(t), q_2(t), p_1(t), p_2(t)) \rightarrow 0 \quad \text{as } t \rightarrow -\infty$$

then $(q_1(t), p_2(t)) \equiv 0$, since the origin is a local center for (15), and this implies $(q_2(t), p_1(t)) \equiv 0$ too.

In spite of this fact, the origin is unstable for (14) for most functions g as above. Theorem 3.3 in [11] proves that stability is equivalent to the isochrony of the periodic solutions of the subsystem (15) in a neighborhood of $0 \in \mathbb{R}^2$, and this implies the isochronous periodicity of all integral curves of (14) in a neighborhood of $0 \in \mathbb{R}^4$. Moreover, Corollary 2.3 in [11] for a smooth g provides

$$(17) \quad g'''(0) = \frac{5g''(0)^2}{3g'(0)}$$

as the simplest necessary condition for (local isochrony and then) stability. So the choice $g(q_1) = q_1 + \sigma q_1^2$ of the Hamiltonian (12) gives instability for all $\sigma \neq 0$. In Figure 2 we can see the projection on the q_1, q_2 -plane of the integral curve of the Hamiltonian vector field given by (12).

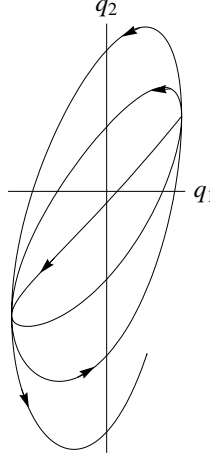


FIGURE 2. Projection of an unbounded orbit for $H = p_1 p_2 + q_1 q_2 + q_1^2 q_2$

Finally, let us remark that the Hamiltonian (12), composed with the symplectic transformation $(Q, P) \mapsto (Q_1 + Q_2, Q_1 - Q_2, P_1 + P_2, P_1 - P_2)/\sqrt{2}$, becomes

$$(18) \quad \frac{1}{2}(Q_1^2 + P_1^2) - \frac{1}{2}(Q_2^2 + P_2^2) + \frac{\sigma}{2\sqrt{2}}(Q_1 + Q_2)(Q_1^2 - Q_2^2)$$

a function with some features in common with Cherry's Hamiltonian (9).

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